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su(n) and sp($2n$) WZW fusion rules

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Abstract. Fusion rules for WZW models based on su(n) and sp($2n$) are considered. Using the results of Kac and Walton it is shown that these fusion rules may be computed using Young diagram methods by performing extra modifications. These modifications take the form of the removal of boundary strips which bear a stiking resemblance to the modifications necessary when performing tensor products for these algebras. This fact is exploited to exhibit a duality between the fusion rules of su(n) at level k and su(k) at level n and also between sp($2n$) at level k and sp($2k$) at level n. The former duality has been discussed in the context of two-dimensional statistical mechanics by Kuniba and Nakanishi and also by Naculich and Schnitzer in the case of WZW models and Goodman and Wenzl for Hecke algebras at roots of unity. For su(3) a manifestly positive combinatorial procedure for computing fusion rules and a generating function for the fusion coefficients are given.

1. Introduction

There has been much interest recently in the classification of rational conformal field theories. These are two-dimensional (2D) conformally invariant theories characterized by the property that the physical Hilbert space $\mathcal{H}$ splits into a finite number of irreducible representations of the chiral algebra $\mathcal{A}_{\mathcal{L}} \oplus \mathcal{A}_{\mathcal{R}}$. The form of this chiral algebra is determined by the particular model, but always contains at least the identity operator together with the left and right copies of the Virasoro algebra. Primary fields are those which create highest weight states of the chiral algebra from the vacuum. Fusion rules are an important aspect of the anaysis of rational conformal field theories, since the fusion coefficients $N_{\lambda\mu\nu}$ are the number of independent couplings of the primary fields λ, μ and ν. In general fusion rules may be calculated by considering the action of modular transformations on characters (Verlinde 1988). For level k WZW models, however, it has recently been shown that the fusion rules may be calculated directly using techniques similar to those used in the calculation of tensor product decompositions of finite-dimensional representations of finite-dimensional, complex, simple Lie algebras (Kac 1989, Walton 1989, Walton 1990). Here we make use of these results to give algorithms for the computation of fusion rules using Young diagrams for models based on su(n) and sp($2n$). A consequence of this formulation is that a natural duality is evident between standard tensor products and fusion rules. That this duality exists

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for su(n) has been discussed by several authors (Goodman and Wenzl 1989, Kuniba and Nakanishi 1990, Naculich and Schnitzer 1990), but that it also holds for sp($2n$) appears to a new observation.

There has also been interest recently in finding expressions for fusion rule coefficients that are explicitly non-negative integers. One would expect that for su(n) fusion rules some variation of the Littlewood–Richardson procedure (Littlewood and Richardson 1934) should exist for calculating fusion coefficients. For su(2) a simple algorithm has been found (Gepner and Witten 1986), but a general formulation is not known. Here we give a manifestly positive algorithm for computing su(3) fusion rules and a generating function for them is derived (see also Cummins *et al* 1990b).

2. Notation and review

Consider $\overline{g}$ a complex, simple, finite-dimensional Lie algebra of rank r and $g \supset \overline{g}$ the untwisted, affine Kac–Moody algebra constructed from $\overline{g}$ as a central extension of the loop algebra of $\overline{g}$. Denote by ω^i, $i = 0 \ldots r$ the fundamental weights of g and $\overline{\omega}^i$, $i = 1 \ldots r$ the fundamental weights of $\overline{g}$. Then any weight λ of g may be written as $\lambda = \sum_{i=0}^r \lambda_i \omega^i$ where $\lambda_i \in Z$ are the Dynkin labels of λ. We define $\overline{\lambda}$, the projection of λ to the weight space of $\overline{g}$, by $\overline{\lambda} = \sum_{i=1}^r \lambda_i \overline{\omega}^i$. The level of λ is given by $\text{level}(\lambda) = \sum_{i=0}^r \lambda_i \check{k}^i$, where $\check{k}^i$ are the comarks of g (for su(n) and sp($2n$) the comarks are all equal to one). For $\overline{\lambda}$ we set $\text{level}(\overline{\lambda}) = \sum_{i=1}^r \lambda_i \check{k}^i$. $P_+ = \{\lambda \mid \lambda_i \geq 0 \;\; i = 0 \ldots r\}$ is the set of integrable highest weights of g and $\overline{P}_+$ is the set of integrable highest weights of $\overline{g}$.

There is a one to one correspondence between the primary fields of level k WZW models and the set $P_+^k = \{\lambda \in P_+ \mid \text{level}(\lambda) = k\}$, or equivalently the set $\overline{P}_+^k = \{\overline{\lambda} \in \overline{P}_+ \mid \text{level}(\overline{\lambda}) \leq k\}$ (Gepner and Witten 1986) (the projection from P_+^k to $\overline{P}_+^k$ is a bijection since given k and $\text{level}(\overline{\lambda})$, λ_0 is uniquely fixed).

Recently it has been pointed out (Kac 1989, Walton 1990) that the fusion rules for these models may be calculated using an 'affinized' Racah–Speiser algorithm (Walton 1989, Weyl 1939, Racah 1964, Speiser 1964). The proof of this algorithm uses the properties of affine Kac–Moody algebras, in particular the modular transformation properties of their characters. It is thus natural from this point of view to state the algorithm in terms of the affine Weyl group acting on weights of g. However, since in this case the affine Weyl group acts only on weights of level $k + \check{h}$ ($\check{h}$ is the dual Coxeter number of $\overline{g}$), it is possible to project onto the weight space of $\overline{g}$. The advantage one gains in doing this is that the algorithm is now precisely that of Racah and Speiser with the Weyl group W of $\overline{g}$ replaced by an affine Weyl group W_k, which is the semidirect product of W with $(k + \check{h})M$. In the cases considered here M is the group of translations in the lattice generated by the long roots of $\overline{g}$ or equivalently the lattice generated by $\{W(\theta)\}$, where θ is the highest weight of the adjoint representation of $\overline{g}$.

Suppose λ and μ are two integrable highest weights of g of level k, then to apply this algorithm we consider the projected weights $\overline{\lambda}$ and $\overline{\mu}$. We apply the Racah–Speiser algorithm as if we were computing the ordinary tensor product of two $\overline{g}$ representations. In this case, however, W is replaced by W_k and rather than reflect the weights into $\overline{P}_+$ they are reflected into $\overline{P}_+^k$. More succinctly, if $N_{\lambda\mu}^{\nu}$ is the multiplicity of the primary field ν in the fusion of λ and μ, and $\overline{N}_{\overline{\lambda}\overline{\mu}}^{\overline{\tau}}$ is the multiplicity of the $\overline{g}$ representation $\overline{\tau}$ in the tensor product of $\overline{\lambda}$ and $\overline{\mu}$, then the above algorithm implies (Kac 1989, Walton 1990)

$$N^{\nu}_{\lambda\mu} = \sum_{\overline{\tau}\in\overline{P}_+} \phi(\overline{\tau})\overline{N}^{\overline{\tau}}_{\overline{\lambda}\overline{\mu}} \tag{1}$$

where

$$\phi(\overline{\tau}) = \begin{cases} \epsilon(w) & \text{if } \overline{\nu} = w(\overline{\tau}+\overline{\rho}) - \overline{\rho}, \text{ for some } w \in W_k \\ 0 & \text{otherwise} \end{cases}$$

$\epsilon(w)$ is the parity of w and $\overline{\rho}$ half the sum of the positive roots of $\overline{g}$.

Young diagram methods exist (Littlewood 1950, King 1971, 1975, Black *et al* 1983) for computing tensor products of representations of classical Lie algebras. An important feature of these methods is that the calculations are performed in two stages. The first involves calculating for infinite rank, usually making use of the Littlewood–Richardson rule in some way. The result for any particular, finite rank is then found by using modification rules. These take the form of the removal of boundary strips from Young diagrams, together with signs depending on the geometry of the removed strip. These modification rules are obtained using certain determinantal expansions of general characters in terms of fundamental characters, together with the observation that the restrictions to finite ranks of fundamental characters are easy to evaluate (Newell 1951, Koike and Terada 1987).

The modification rules for su(n) and sp($2n$) are given in table 1. In this table partition labels for irreducible representations have been used (see Black *et al* 1983 for more details). Note that in addition to these rules, for su(n) any columns of length n may be deleted. The notation $\lambda - h'_1$ denotes the representation label obtained by taking the Young diagram F^λ corresponding to the partition λ and then removing a boundary strip of length h'_1 starting at the bottom of the first *column* and moving up and to the right. If the resulting Young diagram is not regular, i.e. is not the Young diagram of some partition, then the corresponding representation (or more precisely its character) is identically zero. The notation is such that c_1 is the number of columns traversed by the covariant boundary strip (or simply the boundary strip in the case of sp($2n$)) and $\overline{c}_1$ is the number of columns traversed by the contravariant boundary strip. Note that λ' is the partition conjugate to λ, which is the partition obtained by exchanging rows and columns in the corresponding Young diagram F^λ and so λ'_1 is the length of the first column of F^λ. To avoid confusion these modification rules will be referred to as *rank* modification rules. For su(n) there exist two kinds of rank modification rules, one for covariant Young diagrams and the other for mixed Young diagrams, i.e. diagrams representing tensor representations with both covariant and contravariant indices. The modification rules involving boundary strips are applied until the length of the boundary strip to be removed is negative, in which case the diagram is standard, or is zero, in which case the corresponding character is identically zero, or until an irregular Young diagram is produced.

3. Fusion modification rules

We may interpret equation (1) in terms of modification rules. First we calculate the ordinary tensor product of $\overline{\lambda}$ and $\overline{\mu}$. We can think of this step as calculating the fusion

Table 1. The su(n) and sp($2n$) rank modification rules.

Algebra	Modification	Strip length
su(n)	$\{\lambda\} = 0$ if $\lambda_1' \geq n+1$	
	$\{\overline{\mu};\lambda\} = (-1)^{c_1+\bar{c}_1+1}\{\overline{\mu - h_1'};\lambda - h_1'\}$	$h_1' = \mu_1' + \lambda_1' - n - 1 \geq 0$
sp($2n$)	$\langle\lambda\rangle = (-1)^{c_1+1}\langle\lambda - h_1'\rangle$	$h_1' = 2(\lambda_1' - n - 1) \geq 0$

rule for infinite level. The resulting highest weights are in $\overline{P}_+$, but not necessarily in $\overline{P}_+^k$. Suppose $\overline{\tau}$ is a weight in $\overline{P}_+$ that is not in $\overline{P}_+^k$ and that there is a $\overline{\nu} \in \overline{P}_+^k$ and $w \in W_k$ such that $\overline{\nu} = w(\overline{\tau} + \overline{\rho}) - \overline{\rho}$. Then, according to equation (2), $N_{\lambda\mu}^{\nu}$ has a contribution $\overline{N}_{\overline{\lambda}\overline{\mu}}^{\overline{\tau}}$ with the sign given by the parity of w. Thus the problem of computing the fusion coefficients is reduced to finding, given an arbitrary $\overline{\tau}$, a corresponding w and $\overline{\nu}$. Modification rules provide an algorithm for them. The idea is to use relatively simple transformations in W_k the effect on the highest weights of which is easy to interpret combinatorially as the removal of boundary strips from Young diagrams. Each of these 'simple' transformations has the property that the level of the transformed weight is less than the initial weight and lies either in $\overline{P}_+$ or corresponds to a character which is identically zero (this occurs when a irregular Young diagram is produced). Thus iterating the process either produces a weight in $\overline{P}_+^k$ or zero. In principle it is possible after completing this process to construct w, but in fact all we need is $\overline{\nu}$ and the parity of w. The former is produced directly by the algorithm and the latter may be found from the geometry of the strips removed.

Table 2. Representation labels, level, dual Coxeter number and highest weight of adjoint representation for the simple Lie algebras.

Algebra	Representation labels	Level	$\check{h}$	θ
su(n)	$\{\overline{\mu};\lambda\}$	$\mu_1 + \lambda_1$	n	$\{\overline{1},1\}$
sp($2n$)	$\langle\lambda\rangle$	λ_1	$n+1$	$\langle 2\rangle$
$so(2n+1)$	$[\lambda]$	$\lambda_1 + \lambda_2$	$2n-1$	$[1^2]$
	$[\Delta;\lambda]$	$\lambda_1 + \lambda_2 + 1$		
$so(2n)$	$[\lambda]$	$\lambda_1 + \lambda_2$	$2n-2$	$[1^2]$
	$[\lambda]_\pm$	$\lambda_1 + \lambda_2$		
	$[\Delta;\lambda]_\pm$	$\lambda_1 + \lambda_2 + 1$		
e_8	(λ)	λ_1	30	(21^7)
e_7	(λ)	λ_1	18	(21^6)
e_6	$(s:\lambda)$	s	12	$(2:0)$
f_4	(λ)	$\lambda_1 + \lambda_2$	9	(1^2)
	$(\Delta:\lambda)$	$\lambda_1 + \lambda_2 + 1$		
g_2	(λ)	λ_1	4	(21)

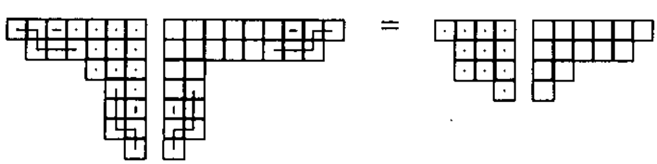
For su(n) and sp($2n$) a modification rule may be constructed in the following way. If a weight has level greater than k (the level of highest weights in partition notation is recorded in table 2; for conventions regarding the labelling of highest weights by partitions see Black *et al* (1983)), then perform a translation by $-(k+\check{h})\theta$. In general this translation results in a weight that no longer lies in $\overline{P}_+$ and it is necessary to apply transformations in W with the twisted action to bring it back into $\overline{P}_+$. It is known that this process can be given a combinatorial interpretation in terms of boundary strip removal, see for example (Black *et al* 1983). The interpretation of this procedure in terms of boundary strip removals for su(n) and sp($2n$) is given in table 3. For sp($2n$) the situation is fairly straightforward, the notation $\lambda - h_1$ means that a boundary strip of length h_1 is removed from the end of the first *row* of the corresponding Young diagram moving downward and to the left, r_1 is the number of rows traversed by the strip. For covariant su(n) diagrams it is necessary to remove a strip from the end of the first row while at the same time adding one of the same length starting at the bottom of the first column at the nth row (assuming that all columns of length n have been initially deleted). This modification rule has been derived by Goodman and Wenzl (1989) in the context of Hecke algebras at roots of unity. It is also possible to state this modification rule for mixed Young diagrams when the addition of a boundary strip to the covariant diagram becomes the removal of a strip from the contravariant diagram. There is a stiking similarity between tables 1 and 3. To obtain the fusion modifications from the rank modifications it is only necessary to replace the rank n by the level k and interchange the roles of rows and columns (i.e. work with transposed Young diagrams), provided for su(n) we consider only modifications of mixed diagrams. In fact if we consider the fusion of a purely covariant with a purely contravariant diagram then only mixed modifications are required, giving rise to a duality which is discussed later. As with standard tensor products the fusion modifications are applied until either the removal of a strip results in an irregular Young diagram, or the strip to be removed has negative length, or until it has length zero. The parity of the element w of W_k may be calculated from the number of rows that the boundary strips traverse, as shown in table 3.

Table 3. The su(n) and sp($2n$) fusion modification rules.

Algebra	Modification	Strip length
su(n)	$\{\lambda\} = (-1)^{r_1+r_n+1}\{\lambda - h_1 + h_n\}$	$h_1 = h_n = \lambda_1 - k - 1 \geq 0$
	$\{\bar{\mu};\lambda\} = (-1)^{r_1+\bar{r}_1+1}\{\overline{\mu - h_1};\lambda - h_1\}$	$h_1 = \mu_1 + \lambda_1 - k - 1 \geq 0$
sp($2n$)	$\langle\lambda\rangle = (-1)^{r_1+1}\langle\lambda - h_1\rangle$	$h_1 = 2(\lambda_1 - k - 1) \geq 0$

3.1. Examples

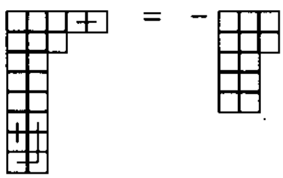
3.1.1. su(8) level 10.

 (2)

Suppose that a representation labeled by the mixed partition $\{\overline{7632^3 1}; 982^4 1\}$ has occurred in some fusion product. Since the lengths of the first columns of the covariant and contravariant diagrams is 7 we have $h_1' = 7 + 7 - 8 - 1 = 5 \geq 0$. So a boundary strip of length 5 must be removed from the covariant and contravariant diagrams. Since both strips traverse two columns the sign factor is $(-1)^{2+2+1} = -1$. This results in the standard diagram $\{\overline{7631^2}; 9821^2\}$. Again, a modification is required since $h_1' = 5 + 5 - 8 - 1 = 1 \geq 0$ and so strips of length 1 must be removed, which results in a further sign factor of -1. No more rank modifications are required since

$\{\overline{7631};9821\}$ has $h'_1 = 4+4-8-1 = -1$. The level, however, is 16 and so fusion modifications are needed. The first strip has length $h_1 = 9+7-10-1 = 5$ and its removal results in the diagram $\{\overline{53^31};7521\}$ and a sign factor -1. Finally it is necessary to remove strips of length 1 yielding $\{\overline{43^21};6521\}$ which has a level of 10. Since each strip removal produces a factor of -1 and there are four such removals, the final sign factor is $+1$. Note that although in principle it is necessary to perform the rank modifications first and then the fusion modifications, in fact in this example we could have reversed the procedure, first removing the strips corresponding to the fusion modifications and then those corresponding to the rank modifications, without changing the final result.

3.1.2. sp(10) level 3.



(3)

The representation labelled by $\langle 532^6\rangle$ is not standard in sp(10), but after two strip removals we obtain $-\langle 532^3\rangle$ which is. This has level 5 and so fusion modifications are required. Removing a strip of length 2 yields $-\langle 3^22^3\rangle$. Again note that the order of the rank and fusion modifications could have been reversed.

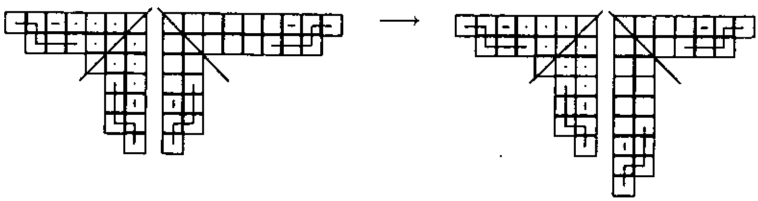
4. Duality

As previously noted, the similarity between fusion modifications and rank modifications gives rise to a duality between the fusion rules for su(n) at level k and su(k) at level n and also sp($2n$) at level k and sp($2k$) at level n. For sp($2n$) this duality is fairly straightforward. Consider the fusion of two sp($2n$) representations $\langle\lambda\rangle$ and $\langle\mu\rangle$ at level k. This may be computed by first finding the standard tensor product of $\langle\lambda\rangle$ and $\langle\mu\rangle$ using Young diagram methods (see Black *et al* 1983) and making use of the rank modification rules of table 1. Finally the fusion modifications of table 3 are applied. The statement of duality is that transposing the resulting diagrams gives the fusion of $\langle\lambda'\rangle$ and $\langle\mu'\rangle$ for sp($2k$) at level n. It is known that this result holds when no modifications are necessary, i.e. for infinite level and rank, thus the only way that this duality could fail would be if fusion and rank modifications did not 'commute'. Fortunately this potential problem does not arise. The reason is that if we start with two integrable highest weights that are standard with regard to rank and level, then the boundary strips that have to be subtracted are fairly short. In fact they cannot reach the 'main diagonal' of the Young diagram. For example for the rank modifications we would have to have $h'_1 \geq \lambda'_1$ or $\lambda'_1 \geq 2n+2$ for the boundary strip to reach the diagonal. But the maximum depth of partition obtained from the product of two standard sp($2n$) diagrams is $2n$. A similar statement holds for the fusion modifications. Thus fusion modifications are restricted to the 'top' of the diagram while rank modifications are limited to the 'bottom' of the diagram, and so the order in which we perform them does not affect the final answer.

The su(n) duality is best seen by considering the fusion of $\{\overline{\lambda}\}$ and $\{\mu\}$ at level k, rather than the product of two covariant diagrams. Here $\{\overline{\lambda}\}$ refers to a contravariant representation of su(n) rather than the projection of a g weight. The statement of duality is that taking the transposes of the diagrams in the fusion of $\{\overline{\lambda}\}$ and $\{\mu\}$ gives the fusion of $\{\overline{\lambda}'\}$ and $\{\mu'\}$ in su(k) at level n, for mixed diagrams the covariant and contravariant diagrams are transposed separately. Once again the potential problem is the ordering of the modifications, and once again this does not occur because the removed strips do not reach the main 'diagonal'. For example in the fusion modifications of the diagram $\{\overline{\sigma};\tau\}$, for the covariant strip to reach the main diagonal we must have $h_1 = \sigma_1 + \tau_1 - k - 1 \geq \tau_1$ or $\sigma_1 \geq k+1$. But it follows from the algorithm for multiplying mixed Young diagrams that σ_1 is at most λ_1, which is less than or equal to k.

4.1. Examples

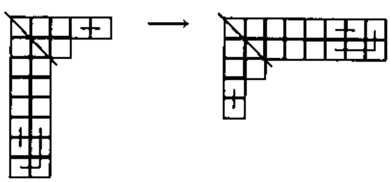
4.1.1. su(8) level 10 and su(10) level(8).



(4)

If the representation $\{\overline{7632^31};982^41\}$ occurs in a fusion calculation for su(8) at level 10, then as we have seen it modifies to $\{\overline{43^21};6521\}$. Duality inplies that there should be a term $\{\overline{43^21};432^31\}$ in the corresponding fusion for su(10) at level 8. The representation $\{\overline{7632^31};762^61\}$ occurs before modifications and the rank (fusion) modifications of this term correspond to the fusion (rank) modifications of $\{\overline{7632^31};982^41\}$ and so the resulting modified diagram is indeed $\{\overline{43^21};432^31\}$.

4.1.2. sp(10) level 3 and sp(6) level 5.



(5)

Similarly the modifications of $\langle 532^6\rangle$ for sp(10) at level 3 are seen to be dual to the modifications of $\langle 8^221^2\rangle$ for sp(6) at level 5. So if a term $\langle 3^22^3\rangle$ occurs in a fusion for sp(10) at level 3 then a term $\langle 5^22\rangle$ occurs in the corresponding fusion for sp(6) at level 5.

5. Fusion rules for su(3)

There is some interest in finding manifestly positive algorithms for computing fusion rules. The only case for which such an algorithm is known for all levels is su(2) (Gepner and Witten 1986). It is possible to find such an algorithm for su(3) using the previous modifications. The key observation is that the rank modifications of a product of a two-rowed covariant diagram with a two-rowed contravariant diagram is trivial in su(3). Any mixed diagrams with a total of four rows are discarded and all other diagrams are standard. This means that the only complications arise from the level modifications, but from duality these are equivalent to products in su(k), and these can be found in a manifestly positive way using the Littlewood–Richardson rule. After a little work this gives rise to the following algorithm:

Input. Level k; $\{\lambda_1, \lambda_2\}$, $\{\sigma_1, \sigma_2\}$ irreducible su(3) representations in partition notation.

Output. Irreducible su(3) representations occurring in the fusion of $\{\lambda_1, \lambda_2\}$ and $\{\sigma_1, \sigma_2\}$ at level k.

Method. Construct all Young tableaux T with the following properties:

(1) T has at most three rows. The number of boxes in row1 $\geq$ the number of boxes in row 2 $\geq$ the number of boxes in row 3.

(2) The boxes of T contain ones or twos or are empty.

(3) The entries of T are weakly increasing across rows (from left to right) and strictly increasing down columns. Empty boxes occur before filled boxes.

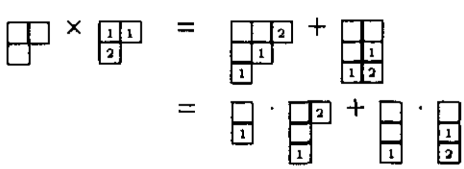
(4) There are λ_1 unfilled boxes in row 1, λ_2 unfilled boxes in row 2 and no unfilled boxes in row 3.

(5) If $a_{i,j}$, $i = 1,2,3$ $j = 1,2$ is the number of entries j in row i then

$$\begin{aligned}
&a_{1,1} + a_{2,1} + a_{3,1} = \sigma_1\\
&a_{1,2} + a_{2,2} + a_{3,2} = \sigma_2\\
&a_{1,1} \leq k - \lambda_1\\
&a_{1,1} + a_{1,2} \leq k - \lambda_1 + a_{3,1}\\
&a_{3,2} \leq a_{2,1}\\
&a_{1,2} + a_{3,2} \leq a_{2,1} + a_{3,1}\\
&\text{if} \quad a_{2,1} < \lambda_1 - \lambda_2 \quad \text{then} \quad a_{2,2} = 0\\
&\text{if} \quad a_{2,1} = \lambda_1 - \lambda_2 \quad \text{then} \quad a_{2,2} \leq a_{1,1}
\end{aligned} \tag{6}$$

(6) Remove from each resulting tableau any columns of length 3. The lengths of the two rows in the final tableaux are the su(3) representation labels.

5.1. Example: level 2



$$= c \cdot g + d \cdot e \tag{7}$$

Here dots denote a stretched product of the elementary tableaux given in table 4, see Cummins *et al* (1990a). We may use the techniques of this paper together with the above algorithm to find a generating function for su(3) fusion rules. The product of the elementary tableaux a and f is forbidden since no twos are allowed in row two below an empty box in row one. There is one other syzygy, namely $dh = cg$; in this case it is necessary to eliminate the product dh. This yields:

$$
\begin{aligned}
G = {} & [(1-z)(1-zL_2N_2)(1-zL_1M_1N_2)(1-zL_1M_2)(1-zL_2M_2N_1)(1-zM_1N_1)]^{-1} \\
& \times \left(1 + \frac{zL_2M_1}{(1-zL_2M_1)(1-zL_1N_1)} + \frac{zM_2N_2}{(1-zM_2N_2)(1-zL_2M_1)}\right. \\
& + \frac{zL_1N_1}{(1-zL_1N_1)(1-z^2L_1M_2N_1N_2)} \\
& \left. + \frac{z^2L_1M_2N_1N_2}{(1-z^2L_1M_2N_1N_2)(1-zM_2N_2)}\right)
\end{aligned}
\tag{8}
$$

which simplifies to (Cummins *et al* 1990b):

$$
\begin{aligned}
G = {} & [(1-z)(1-zL_1N_1)(1-zL_2N_2)(1-zL_2M_1) \\
& \times (1-zL_1M_2)(1-zM_1N_1)(1-zM_2N_2)]^{-1} \\
& \times \left(\frac{1-z^3L_1L_2M_1M_2N_1N_2}{(1-zL_1M_1N_2)(1-zL_2M_2N_1)}\right).
\end{aligned}
\tag{9}
$$

Table 4. Elementary tableaux for su(3)fusion rules. Auxiliary labels carry Dynkin labels as their exponents.

	Auxiliary Labels	Tableau		Auxiliary Labels	Tableau		Auxiliary Labels	Tableau
a	L_1N_1	$\ydiagram{1}$	d	L_2M_1	$\ytableaushort{{},{},1}$	g	$L_2M_2N_1$	$\ytableaushort{{}2,{},1}$
b	L_2N_2	$\ydiagram{1,1}$	e	L_1M_2	$\ytableaushort{{},1,2}$	h	$L_1M_2N_1N_2$	$\ytableaushort{{}2,1}$
c	$L_1M_1N_2$	$\ytableaushort{{},1}$	f	M_2N_2	$\ytableaushort{1,2}$	i	M_1N_1	$\ytableaushort{1}$

The coefficient of $z^k L_1^{a_1} L_2^{a_2} M_1^{b_1} M_2^{b_2} N_1^{c_1} N_2^{c_2}$ in the Taylor series expansion of G is the multiplicity of the representation with Dynkin labels (c_1, c_2) in the fusion at level k of (a_1, a_2) with (b_1, b_2), again these are Dynkin labels.

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